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Some Fixed Point Results in Ordered *G*-Metric Spaces

Komal Goyal[1] and Bhagwati Prasad[2]

[1,2]Department of Mathematics Jaypee Institute of Information Technology A-10, Sector-62, Noida, UP-201307 INDIA
E-mail: [1]komal.goyal0988@gmail.com, [2]b_prasad10@yahoo.com

Abstract—*The study of fixed points of maps satisfying certain contractive conditions in varieties of spaces has been emerged as an important area of research in field of fixed point theory and its applications. The purpose of this paper is to obtain a fixed point theorem for maps satisfying some rational type contractive condition in the framework of ordered G-metric spaces. Our results improve and extend some of the recent results reported in the literature.*

1. INTRODUCTION

After the celebrated Banach contraction principle (BCP) in 1922, there have been numerous results in the literature of fixed point theory dealing with mappings satisfying the contractive conditions of nonlinear types as well. In 1968, there was another important development in fixed point theory when Kannan [7] proved a fixed point theorem for the maps not necessarily continuous. Subsequently, a number of authors such as Hardy and Rogers [6] Reich [19], Suzuki [28] and many others have obtained interesting generalizations of BCP and various results pertaining to fixed points, approximate fixed point, common fixed points, coincidence points, etc. have been established for maps satisfying contractive conditions in the settings of different spaces. Rhoades [20] is an appropriate reference for a fundamental comparison and development of various contractive conditions (see also [1-5], [8-18], [21-28] and several references thereof). Thus fixed point theory has been extensively studied, generalized and enriched in different approaches specially, metric, topological and order-theoretic. This advancement in fixed point theory diversified the applications of various fixed point results in various areas such as fractal and chaos theory, dynamical systems, existence theory of ODE, system analysis, dynamic programming, optimization and game theory, information and control theory and other diverse disciplines of mathematics, science and engineering.

The concept of 2-metric space was introduced by Gahler [4-5] as a generalization of usual notion of metric space. In 1992, Dhage [2] extended this metric space and introduced *D*-metric space. Several authors proved the existence and uniqueness of a fixed point of contractive mapping in the context of *D*-metric space. Unfortunately, in 2003-2004, Mustafa and Sims [11] noticed that most of the claims made in *D*-metric spaces were incorrect. These facts determined them to introduce a new concept called *G*-metric space (also see [12-13]). Recently, Saadati et al. [22] obtained some fixed point result for contractive mappings in partially ordered *G*-metric spaces.

The aim of this paper is to establish a fixed point theorem for maps satisfying some rational type contractive condition in the framework of ordered *G*-metric spaces.

2. PRELIMINARIES

In this section we present the basic concepts and relevant results required in the sequel.

Definition 2.1 [11]. Let X be a nonempty set. Suppose that a mapping $G: X \times X \times X \to R^+$ satisfies

(G_1) $G(x,y,z) = 0$ if $x = y = z$;

(G_2) $0 < G(x,y,z)$ for all $x, y, z \epsilon X$, with $x \neq y$;

(G_3) $G(x,x,y) \leq G(x,y,z)$ for all $x, y, z \epsilon X$, with $y \neq z$;

(G_4) $G(x,y,z) = G(x,z,y) = G(y,z,x) = \cdots$.. (symmetry in all three variables) ;

(G_5) $G(x,y,z) \leq G(x,a,a) + G(a,y,z)$ for all $x, y, z, a \epsilon X$.

Then, G is called a G-metric on X and (X, G) is called a G-metric space.

Definition 2.2 [12]. A sequence $\{x_n\}$ in a G-metric space X is

(i) A G-Cauchy sequence if, for every $\varepsilon > 0$, there is a natural number n_0 such that for all $n, m, l \geq n_0$, $G(x_n, x_m, x_l) < \varepsilon$,

(ii) A G-Convergent sequence if, for any $\varepsilon > 0$, there is an $x \epsilon X$ and an $n_0 \epsilon N$, such that for all $n, m \geq n_0$, $G(x_n, x_m, x) < \varepsilon$.

A G-metric space on X is said to be G-complete if every G-Cauchy sequence in X is G-convergent in X. It is known that

$\{x_n\}$ G-converges to $x \epsilon X$ if and only if $G(x_n, x_m, x) \to 0\ as\ n, m \to \infty$.

Proposition 2.1 [12]. Let X be a G-metric space. Then, the following are equivalent.

(i) The sequence $\{x_n\}$ is G-convergent to x.

(ii) $G(x_n, x_n, x) \to 0\ as\ n \to \infty$

(iii) $G(x_n, x, x) \to 0\ as\ n \to \infty$

(iv) $G(x_n, x_m, x) \to 0\ as\ n, m \to \infty$

Proposition 2.2 [12]. Let X be a G-metric space. Then, the following are equivalent.

(i) The sequence $\{x_n\}$ is G-Cauchy.

(ii) For every $\varepsilon > 0$, there exists $n_0 \epsilon N$, such that for all $n, m \geq n_0, G(x_n, x_m, x_m) < \varepsilon$; that is, if $G(x_n, x_m, x_m) \to 0\ as\ n, m \to \infty$.

3. MAIN RESULTS

In this section, we present our main result in ordered complete G-metric space.

Theorem 3.1. Let (X, G) be a complete G-metric space and $T : X \to X$ satisfy the following condition:

$$G(Tx, Ty, Ty) \leq \left(\frac{G(x,Ty,Ty)+G(y,Tx,Tx)}{G(x,Tx,Tx)+G(y,Ty,Ty)+1}\right) G(x, y, y), \tag{3.1}$$

for all $x, y \in X$. Then,

(i) T has at least one fixed point $x \in X$.

(ii) $\{T^n x\}$ converges to a fixed point, for all $x \in X$.

(ii) If x^*, y^* are two distinct fixed point of T, then $G(x^*, y^*, y^*) \geq \frac{1}{2}$.

Proof. Let $x_0$be any arbitrary element of X. Construct a sequence $\{x_n\}$ such that $x_{n+1} = Tx_{n.}$ for n =0, 1, 2, …, n.

We have

$$G(x_{n+1}, x_n, x_n) = G(Tx_n, Tx_{n-1}, Tx_{n-1}) \leq$$

$$\left(\frac{G(x_n, x_n, x_n) + G(x_{n-1}, x_{n+1}, x_{n+1})}{G(x_n, x_{n+1}, x_{n+1}) + G(x_{n-1}, x_n, x_n) + 1}\right) G(x_n, x_{n-1}, x_{n-1})$$

$$\leq$$

$$\left(\frac{G(x_{n-1},x_n,x_n)+G(x_n,x_{n+1},x_{n+1})}{G(x_n,x_{n+1},x_{n+1})+G(x_{n-1},x_n,x_n)+1}\right) G(x_n, x_{n-1}, x_{n-1}) \tag{2}$$

Given

$$\beta_n = \left(\frac{G(x_{n-1},x_n,x_n)+G(x_n,x_{n+1},x_{n+1})}{G(x_n,x_{n+1},x_{n+1})+G(x_{n-1},x_n,x_n)+1}\right) \tag{3.2}$$

We have,

$$G(x_{n+1}, x_n, x_n)$$
$$\leq \beta_n G(x_n, x_{n-1}, x_{n-1})$$
$$\leq \beta_n \beta_{n-1} G(x_{n-1}, x_{n-2}, x_{n-2})\ .$$
$$\leq \beta_n \beta_{n-1 \ldots\ldots} \beta_1 G(x_1, x_0, x_0)$$

Observe that the sequence $\{\beta_n\}$ is non increasing, with positive terms, so we have

$$\beta_1 \beta_2 \ldots\ldots \beta_n \leq \beta_1^n \text{ and } \beta_1^n \to 0.$$

It follows that

$$\lim_{n\to\infty}(\beta_1 \beta_2 \ldots\ldots \beta_n) = 0. \tag{3.3}$$

Thus, it is verified that

$$\lim_{n\to\infty}(G(x_{n+1}, x_n, x_n)) = 0. \tag{3.4}$$

Now for all $m, n \in N$ we have

$$G(x_m, x_n, x_n) \leq G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2}) + \cdots + G(x_{m-1}, x_m, x_m)$$
$$\leq [(\beta_n \beta_{n-1} \ldots \beta_1) + (\beta_{n+1} \beta_n \ldots \beta_1) + \cdots + (\beta_{m-1} \beta_{m-2} \ldots \beta_1)] G(x_1, x_0, x_0)$$

$$= \sum_{k=n}^{m-1} (\beta_k \beta_{k-1} \ldots \beta_1)\, G(x_1, x_0, x_0)$$

Suppose that $a_k = (\beta_k \beta_{k-1} \ldots \beta_1)$.
Since

$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = 0. \tag{3.5}$$

Therefore,

$$\sum_{k=1}^{\infty} a_k < \infty.$$

It means that, as $m, n \to \infty$ we have

$$\sum_{k=n}^{m-1} (\beta_k \beta_{k-1} \ldots \beta_1) \to 0, \tag{3.6}$$

In other words,$\{x_n\}$ is a Cauchy sequence and so converges to $x^* \in X$.

We claim that x^* is a fixed point.

Note that

$$G(Tx_n, Tx^*, Tx^*) \leq \frac{G(x_n,Tx^*,Tx^*)+G(x^*,Tx_n,Tx_n)}{G(x_n,Tx_n,Tx_n)+G(x^*,Tx^*,Tx^*)+1} G(x_n, x^*, x^*) \tag{3.7}$$

On taking limit on both sides of (3.7), we have

$$G(x^*, Tx^*, Tx^*) = 0.$$

Thus,

$$Tx^* = x^*$$

If there exists two distinct fixed points $x^*, y^* \in X$ .Then,

$$G(x^*, y^*, y^*) = G(Tx^*, Ty^*, Ty^*)$$

$$\leq \left[\frac{G(x^*,y^*,y^*)+G(y^*,x^*,x^*)}{(G(x^*,x^*,x^*)+G(y^*,y^*,y^*)+1)}\right] G(x^*, y^*, y^*) \tag{3.8}$$
$$= 2[G(x^*, y^*, y^*)]^2$$

Therefore, $G(x^*, y^*, y^*) \geq \frac{1}{2}$.

This completes the proof.

To illustrate the result of above theorem, we present following example.

Example 3.1. Let $X = [0, 1]$ with G-metric defined on X as follows:

$$G(x, y, z) = \max\{|x - y|, |y - z|, |z - x|.$$

Let $f: X \to X$ be defined by

$f(x) = \frac{3}{4}x$ for all $x \epsilon X$.

We have,

$$G(x, Tx, Tx) = \left|x - \frac{3}{4}x\right| = \frac{1}{4}x$$

$$G(y, Ty, Ty) = \left|y - \frac{3}{4}y\right| = \frac{1}{4}y$$

$$G(Tx, Ty, Ty) = \left|\frac{3}{4}(x - y)\right|$$

$$G(x, Ty, Ty) = \left|x - \frac{3}{4}y)\right|$$

$$G(y, Tx, Tx) = \left|y - \frac{3}{4}x)\right|$$

$G(x, y, y) = |x - y)|$.

So that,

$$G(Tx, Ty, Ty) \leq \left(\frac{G(x,Ty,Ty)+G(y,Tx,Tx)}{G(x,Tx,Tx)+G(y,Ty,Ty)+1}\right) G(x, y, y).$$

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